Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 25, N. 1, 57-76

Biasymptotic Solutions of Perturbed Integrable Hamiltonian Systems

L. H. Eliasson

Abstract. We prove that small perturbations of a real analytic integrable Hamiltonian system in d degrees of freedom generically have biasymptotic orbits which are obtained as intersections of the stable and unstable manifolds of invariant hyperbolic tori of dimension $d-1$. Hence, these solutions will be forward and backward asymptotic to such a torus and not to a periodic solution. The generic condition, which is open and dense, is given by an explicit condition on the averaged perturbation.

1. Introduction

A perturbation of an integrable Hamiltonian system has generically an invariant hyperbolic torus. This torus, which does not exist in the unperturbed system, is present in the averaged system (for a generic perturbation) and its persistence to the perturbed system follows from general results on conservation of invariant hyperbolic tori as is shown in [**1,2,3**] – see also [**4**] for a somewhat different approach using centermanifolds. In general we don't know anything about the global behavior of the stable and unstable manifolds, not even for the averaged system. Some conditions which assure the intersection of these manifolds for hyperbolic periodic orbits have been obtained by variational methods [**5**] but these do not apply, at least not in any immediate way, to our problem.

The situation is different, however, when the hyperbolic torus is of dimension $d-1$ because then the averaged system is integrable and global information is available. In the non-degenerate situation – i.e. the hyperbolicity does not go to zero with the perturbation – the intersection, and splitting, and transversality, of the stable and unstable manifolds

Received 8 March 1993.

is a pure perturbation problem. For a sufficiently small perturbation of such a system the intersection, and splitting, and transversality can be detected by a Poincaré-Melnikov-integral – [**6**,**7**,**8**]. Our problem however is degenerate since the averaged system depends on the perturbation parameter – in particular the hyperbolicity of the averaged system goes to 0 with the perturbation – and therefore the problem is not purely perturbative. This phenomenon is often referred to as "exponentially small splitting" and is described in for example [**9**].

The degenerate problem that we have described can be handled for symplectic mappings in the plane by an argument using area-preservation originally due to Poincaré – see for example [**10**]. Instead of area-preservation, which makes no sense in our problem, we shall exploit the exactness of the Hamiltonian flow. A general formulation of such an intersection property has been given in [**11**]. Using these more global ideas we shall prove the intersection of the stable and unstable manifolds along at least d different orbits – this will hold under the only condition that the averaged system has a hyperbolic structure.

Statement of theorem A

Consider in the symplectic space $(\mathbf{T}^d \times \mathbb{R}^d, \Sigma dx_i \wedge dy_i)$ the integrable Hamiltonian vector field $J\nabla H_o$ given by the analytic function

$$H_o(y) = \langle \omega, y\rangle + \langle My, y\rangle + O^3(y). \tag{1}$$

defined near $y = 0$ in $\mathbb{R}^d$, where M is an antisymmetric matrix and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$.

The torus $y = 0$ is invariant and we shall assume that it is a resonant torus with a single resonance but otherwise diophantine:

$$\begin{aligned} &\langle n_o, \omega\rangle = 0, \\ &|\langle n, \omega\rangle| \geq K|n|^{-\tau}, n \in \mathbb{Z}^d \backslash \{n_o\mathbb{Z}\}, \end{aligned} \tag{2}$$

for some $\tau > d-2$, $K > 0$. The non-degeneracy condition is

$$\det M \neq 0 \text{ and } \langle Mn_o, n_o\rangle \neq 0 \text{ positive say.} \tag{3}$$

This system has no hyperbolicity at all so we must get it from the perturbation. Let $H(x, y)$ be analytic and let

$$f(x) = f_H(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T H(x + t\omega, 0) dt.$$

f is analytic and invariant under $x \to x + t\omega$, $t \in \mathbb{R}$, and we shall assume that

$$\begin{aligned} &f \text{ has a unique maximum } (\operatorname{mod} \mathbb{R}\omega) \text{ at some point } p, \\ &f''(p) \text{ has a strictly negative eigenvalue.} \end{aligned} \tag{4}$$

Theorem A. *Under the above assumptions there exists a positive constant ε_o – depending on H_o and H – such that if $\varepsilon^2 < \varepsilon_o$, then the system $H_o + \varepsilon^2 H$ has an invariant hyperbolic torus whose induced vector field is analytically conjugate to a Kronecker flow with frequency vector ω and whose stable and unstable manifolds intersect along at least d different orbits.*

We have made a certain number of choices of signs in the formulation of this theorem: of $\langle Mn_o, n_o \rangle$; of the perturbation parameter ε^2; of the sign of the non-zero eigenvalue of $f''(p)$. They can be made differently but what is important is that the sign of $\langle Mn_o, n_o \rangle$ and the sign of the non-zero eigenvalue of the averaged perturbation (i.e. of $\varepsilon^2 f''(p)$) are different.

In the case of non-autonomous quasi-periodic perturbations of one degree of freedom Hamiltonians we have $n_o = (1, 0, \dots, 0)$ and $\langle My, y \rangle = M_1 y_1^2$. Clearly $\det M = 0$ but since the perturbation does not depend on $(y_2, \dots, y_d)$, the theorem holds also in this case.

The proof of this theorem will be done by reducing it to theorem B described below.

Statement of theorem B

Consider instead the Hamiltonian system given by

$$H_o(x, y) = \langle \omega, y' \rangle + \frac{\beta^2}{2} y_1^2 + f(x_1) + 2\langle m, y' \rangle y_1 + \langle My', y' \rangle, \tag{5}$$

where $x = (x_1, x') \in \mathbf{T}^d$, $y = (y_1, y') \in \mathbb{R}^d$.

We shall assume that

$$|\langle n, \omega \rangle| \geq K |n|^{-\tau}, \forall n \in \mathbb{Z}^{d-1} \backslash \{0\}, \tag{6}$$

for some $\tau > d-2$, $K > 0$. We also need a non-degeneracy condition:

$$\det \begin{pmatrix} \frac{\beta^2}{2} & m \\ m^* & M \end{pmatrix} = \gamma \neq 0 \text{ and } \beta > 0. \tag{7}$$

The function f should have a unique and non-degenerate global maximum at some point, and satisfy a smoothness condition. We require that it is analytic and

$$\begin{aligned} & f(0) = f'(0) = 0 \text{ and } f''(0) = -\alpha^2,\ \alpha > 0, \\ & \delta = \max\{f(x_1)\colon x_1 \text{ critical point } \neq 0\} < 0, \\ & \sup_{|\operatorname{Im} x_1| < r} |f(x_1)| \leq 1. \end{aligned} \tag{8}$$

So the Hamiltonian vector field $J\nabla H_o$ has a hyperbolic invariant torus of dimension $d-1$ at $x_1 = y_1 = 0$, $y' = 0$. The separatrices are given by

$$y_1 = \pm\frac{1}{\beta}\sqrt{-2f(x_1)}.$$

Theorem B. *Suppose H_o satisfies (5–8). Then there exists a constant ε_1 such that if the analytic function H satisfies*

$$\begin{aligned} & |H|_D = \sup_D |H| < \varepsilon_1, \\ & D = D(r,s) = \{(x,y)\colon |\operatorname{Im} x| < r, |y| < s\},\ s > \frac{2}{\beta}, \end{aligned}$$

then the system $H_o + H$ has an invariant hyperbolic torus whose induced vector field is analytically conjugate to a Kronecker flow with frequency vector ω and whose stable and unstable manifolds intersect along at least d different orbits.

The constant ε_1 depends only on $|H_o - \langle\omega,\rangle|_D, \min(r,s), \alpha, \beta, \gamma, \delta$ and the diophantine constants K, τ.

The assumption on the domain, i.e. $s > \frac{2}{\beta}$, serves only to assure that the perturbed Hamiltonian is defined in a neighbourhood of the separatrices.

The important point, which permits the deduction of theorem A from theorem B, is the independence of ε_1 on the norm of ω when this norm becomes large. If we permitted ε_1 to depend on $|\omega|$ then the intersection would be a pure perturbation problem which could be detected by the variational equation. In our case we need a more global argument.

Even though exactness is not mentioned in the theorem it plays an essential role in the proof so we will recall the definition and some elementary properties.

Exact symplectic transformations

Consider the one-form $\theta = -\Sigma y_i dx_i$. A mapping χ is symplectic if $\chi^*\theta - \theta$ is closed and it is exact symplectic if this one-form is exact, i.e. $= d\phi$ for some function ϕ. Since phase space is not simply connected there are closed one-forms that are not exact, and there are symplectic mappings that are not exact. However, the flow map χ_t of a Hamiltonian system $J\nabla H$ is always exact because

$$\begin{aligned}\chi_t^*\theta - \theta &= \int_0^t \frac{d}{ds}(\chi_s^*\theta)ds = \int_0^t \chi_s^*(d\theta\rfloor J\nabla H + d(\theta\rfloor J\nabla H))ds \\ &= d\int_0^t \chi_s^*(H + \theta\rfloor J\nabla H)ds.\end{aligned}$$

One verifies easily that a composition of two exact symplectic transformations is an exact symplectic transformation.

The important property that we shall use is that an exact transformation which is C^1-close to the identity has a generating function. Suppose $(\tilde{x}, \tilde{y}) = \chi(x, y)$ and let $\tilde{y}d\tilde{x} - ydx = d\phi(x, y)$. Let

$$\psi(x, y) = \phi(x, y) - y(\tilde{x} - x)$$

and consider ψ as a function of $\tilde{x}, y$ through

$$\tilde{x} = \chi_1(x, y) \Longleftrightarrow x = \varphi(\tilde{x}, y),$$

which is possible because χ is C^1-close to the identity. Then

$$\begin{aligned}\tilde{x} &= x - \frac{\partial\psi}{\partial y}, \\ \tilde{y} &= y + \frac{\partial\psi}{\partial\tilde{x}}.\end{aligned}$$

Outline of the paper

The proof will proceed along the following lines. In section 2 we reduce theorem A to theorem B and in the rest of the paper we shall prove theorem B.

In section 3 we formulate a local result – proposition 1 – which gives the invariant torus and the local asymptotic manifolds in exact symplectic coordinates. We then follow up these local coordinates along the unstable manifolds. In doing this we have to compare the time-T-maps of the averaged system and the perturbed system for some fixed time T, independent of ε and ω. The fact that these two vector fields may be large, due to the factor ω, is of no importance and these two mappings are ε-close, independent of $|\omega|$, in the C^1-norm. We then obtain an exact symplectic mapping which is arbitrarily C^1-close to the identity as $\varepsilon \to 0$ and which takes (a part of) the local stable manifold into the global unstable manifold. Using a generating function we then reduce the intersection problem to the existence of critical points of a C^1-function on $\mathbf{T}^{d-1}$.

In section 4 we reduce proposition 1 to a perturbation result on normal forms near an invariant hyperbolic torus – proposition 2. This normal form is given in exact coordinates and is somewhat more refined than we have seen elsewhere so we give a rather brief proof of it in section 5. The proof differs in no essential way from that in for example **[3]**.

2. Theorem B Implies Theorem A

Consider now $H_o + \varepsilon^2 H$ satisfying (1–4). We shall transform it to a Hamiltonian satisfying the setup (5–8) of theorem B.

First of all we can assume that $n_o = e_1 = (1, 0, \dots, 0)^*$, i.e. that the frequency vector is of the form $(0, \omega)$ with $\omega \in \mathbb{R}^{d-1}$ diophantine. In order to see this we choose $C \in Sl(d, \mathbb{Z})$ such that $Cn_o = e_1 = (1, 0, \dots, 0)^*$ – such a matrix always exists since n_o is primitive, i.e.

$\langle n, \omega \rangle = 0$ implies that $n \in n_o\mathbb{Z}$ [**12**]. Then the change of variables

$$(x, y) \longmapsto (C^*x, C^{-1}y)$$

is defined on $\mathbf{T}^d \times \mathbb{R}^d$ and transforms H_o to a function of the same form satisfying (1–3) with $n_o = e_1 - M$ and K will of course depend on the choice of C. The function f gets transformed to the corresponding function for the transformed system, still satisfying (4), and becomes a function of x_1 only.

Secondly we can assume that f attains it maximum at $x_1 = 0$ and that this maximum is 0.

Now $H_o + \varepsilon^2 H$ will be defined on some complex domain

$$D = D(r, s) = \{(x, y) \colon |\mathrm{Im}\, x| < r, |y| < s\}.$$

and we can assume that $K, r, s < 1$, and $|H_o|_D =< 1$.

An averaging transformation

We shall now construct a solution $S = S(x), R = R(x, y) \in \mathbf{O}(y)$ of the equation

$$\{S, H_o\} = H - f - R,$$

where $\{\ ,\ \}$ denotes the Poisson bracket with respect to the symplectic form. Expressed in Fourier coefficients the equation becomes

$$i\langle n, \omega \rangle \hat{S}_{k,n} = \hat{H}_{k,n}(0), \qquad k \in \mathbb{Z}, n \in \mathbb{Z}^{d-1} \backslash 0,$$
$$\{S, H_o - \langle \omega, \cdot \rangle\} = H(x, y) - H(x, 0) - R.$$

Standard estimates of small divisors (see [**13**]) together with Cauchy estimates of derivatives give

$$|S|_{D(r-\rho,s)} < c\frac{1}{Kp^{\tau+1}}|H|_D, \qquad |R|_{D(r-2\rho,s-\sigma)} < c\frac{1}{K\rho^{\tau+2}\sigma}|H|_D.$$

The constant c only depends on τ. In the sequel we shall denote all constants that only depend on τ by the same letter c – the dependence on d is controlled by the dependence on τ, since $\tau > d - 2$.

The time-t-map $\Phi^t, 0 \le t \le 1$, of $\varepsilon^2 J\nabla S$ preserves x and maps

$$D(r - 3\rho, s - 3\sigma) \to D(r - 3\rho, s - 2\sigma),$$

if just $\varepsilon^2 < cK\rho^{\tau+2}\sigma$.

We now transform the system $H_o + \varepsilon^2 H$ by the map Φ^1.

$$\begin{aligned}(H_o + \varepsilon^2 H) \circ \Phi^1 &- (H_o + \varepsilon^2(f + R)) = \\ &= \int_0^1 \frac{d}{dt}(H_o + \varepsilon^2 tH + \varepsilon^2(1-t)(f+R)) \circ \Phi^t dt \\ &= \varepsilon^4 \int_0^1 \{tH + (1-t)(f+R), S\} \circ \Phi^t dt = \varepsilon^4 H_1.\end{aligned}$$

and an easy estimate of the error gives

$$|H_1|_{D(r-3\rho, s-3\sigma)} < c(K\rho^{\tau+2}\sigma)^{-2}.$$

If we now let $6\rho = r$ and $6\sigma = s$, then the transformed system $H_o + \varepsilon^2(f+R) + \varepsilon^4 H_1$ is defined on $D\left(\frac{r}{2}, \frac{s}{2}\right)$.

A scaling

We now do a scaling, replacing y by εy and dividing the system by ε^2. The transformed system is Hamiltonian and given by an integrable part

$$\langle \frac{\omega}{\varepsilon}, y' \rangle + \langle My, y \rangle + f(x_1), \ y = (y_1, y'),$$

which satisfies (5–8), plus a perturbation term $H_2(x, y, \varepsilon)$ which consists of H_1, R and terms of order ≥ 3 in H_o, and which is defined on $D\left(\frac{r}{2}, \frac{s}{2\varepsilon}\right)$.

If $\varepsilon < \frac{1}{4}\beta s$, $\beta^2 = 2\langle Me_1, e_1 \rangle$, then

$$D\left(\frac{r}{2}, \frac{s}{2\varepsilon}\right) \supset D\left(\frac{r}{2}, \frac{2}{\beta}\right)$$

and on the domain $D\left(\frac{r}{2}, \frac{2}{\beta}\right)$ we have

$$|H_2(x,y)| < c(\varepsilon^2(Kr^{\tau+2}s)^{-2} + \varepsilon(Kr^{\tau+2}s^2\beta)^{-1} + \varepsilon(\beta s)^{-3}.$$

Hence all conditions of theorem B are fulfilled and this proves theorem A.

3. Reduction to a Local Problem

Consider now H_o as given by (5–8) and let's write (x, y, ξ, η) for (x_1, y_1, x', y'). Let

$$B(r) = \{(x, y, \xi, \eta) : |x|, |y|, |\eta|, |\mathrm{Im}\, \xi| < r\}.$$

If now $r < s$, as we can assume, then H_o is defined on $B(r)$ and we can use the following proposition.

Proposition 1. *There exists a constant ε_2 such that if*

$$|H|_{B(r)} < \varepsilon < \varepsilon_2,$$

then there exist constants a, b and an exact symplectic transformation $\Phi: B(Rr) \to B(r)$ such that

$$(H_o + H) \circ \Phi = \text{const.} + \langle \omega, \eta - a \rangle + bh(x, y + \frac{2}{\beta^2}\langle m, \eta \rangle) +$$
$$+ \mathbf{O}^2(h(x, y + \frac{2}{\beta^2}\langle m, \eta \rangle), \eta - a), \ h(x, y) = \frac{\beta^2}{2}y^2 + f(x),$$

and

$$|\Phi - id|_{B(R)} + |a| + |b - 1| < A\varepsilon.$$

The constants ε_2 and A depend only on $|H_o - \langle w, \rangle|_{B(r)}$, r, α, β, γ, K, τ, and R only depends on $|H_o - \langle w, \rangle|_{B(r)}, r, \alpha, \beta$.

We shall arrange these coordinates somewhat more using the following lemma which can be proven by a power series expansion but which is also a particularly simple case of a theorem of Rüssmann [**14**].

Lemma. *There exist an exact symplectic transformation φ, defined in a neighbourhood of the origin, and an analytic function g such that $h(\varphi(x, y)) = g(xy)$, with $\varphi(0) = 0$ and*

$$\varphi'(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{\beta}{\alpha}} & -\sqrt{\frac{\beta}{\alpha}} \\ \sqrt{\frac{\alpha}{\beta}} & \sqrt{\frac{\alpha}{\beta}} \end{pmatrix}.$$

Let now

$$\Phi_1: (x, y, \xi, \eta) \to (x, y - \frac{2}{\beta^2}\langle m, \eta \rangle, \xi + \frac{2}{\beta^2}xm, \eta),$$
$$\Phi_2: (x, y, \xi, \eta) \to (\varphi(x, y), \xi, \eta).$$

$\tilde{\Phi} = \Phi_1 \circ \Phi_2$ is exact symplectic and satisfies

$$\tilde{\Phi}(B(R_1 s)) \subset B(s) \text{ and } \tilde{\Phi}^{-1}(B(R_2 s)) \subset B(s), \quad \forall s < r,$$

where R_1 and R_2 only depend on $|H_o - \langle \omega, \rangle|_{B(r)}, r, \alpha, \beta$. Moreover

$$(H_o + H) \circ \Phi \circ \tilde{\Phi} = \text{ const. } + \langle \omega, \eta - a \rangle + bg(xy) + \mathbf{O}^2(g(xy), \eta - a).$$

Proof of theorem A. For simplicity we shall assume that the irrelevant constant in $(H_o+H)\circ\Phi$ is zero, thus saving a letter. We can also assume that $r<s$ so that $D(r,s)\supset B(r)$ – otherwise we have to replace r by $\min(r,s)$. If now

$$|H|_{D(r,s)}<\varepsilon<\varepsilon_2,$$

then the existence of the invariant hyperbolic torus and its two asymptotic manifolds follow from proposition 1 so we must only consider the intersection.

Let H_N be

$$\langle\omega,\eta-a\rangle+bh(x,y+\frac{2}{\beta^2}\langle m,a\rangle)+2b\langle m,\eta-a\rangle(y+\frac{2}{\beta^2}\langle m,a\rangle)+ \\ +b\langle M(\eta-a),(\eta-a)\rangle$$

The invariant torus and its local asymptotic manifolds, as well as the dynamics on them, are the same for H_N and $(H_o+H)\circ\Phi$ but H_N is a global Hamiltonian, defined everywhere with the only restriction that $|\operatorname{Im}x|<r$. The vector field $J\nabla H_N$ equals a constant part $J\nabla\langle\omega,\eta-a\rangle$ plus a part which is essentially independent on ω – it depends on ω only through a and b. If χ_t^N is the flow map of $J\nabla H_N$ an easy estimate of the variational equation gives that

$$\left|\chi_t^N(z')-\chi_t^N(z)\right|<\epsilon^{A_1|bt|}|z'-z|.$$

if $\operatorname{Im}z=0$ and $|\operatorname{Im}z'|<\epsilon^{-A_1|bt|}r$, where z is shorthand for (x,y,ξ,η). The constant A_1, as well as the following constants, depend on the same parameters as A.

In the same way we get for the flow map ξ_t of $J\nabla(H_o+H)$ that if $z\in D(r,s)$ then

$$\left|\chi_t(z)-\chi_t^N(z)\right|<A_2\epsilon^{A_1|t|}\varepsilon,$$

as long as the two flows remain in $D(r,s)$.

Consider now two ξ-sections

$$\begin{aligned}&(x_1,y_1,\xi,a)\in B(Rr)\\&x_1>1\\&h(x_1,y_1+\frac{2}{\beta^2}\langle m,a\rangle)=0\end{aligned}$$

and

$$\begin{aligned}&(x_2, y_2, \xi, a) \in B(Rr)\\ &x_2 < 1\\ &h(x_2, y_2 + \frac{2}{\beta^2}\langle m, a\rangle) = 0.\end{aligned}$$

There is a positive number T such that for all ξ

$$\chi_T^N(x_1, y_1, \xi, a) = (x_2, y_2, \tilde{\xi}, a),$$

where $\tilde{\xi}$ depends on ξ. The number T is easy to compute:

$$T = \frac{1}{b\beta}\int_{x_1}^{x^2} \frac{1}{\sqrt{-2f(u)}} du < \frac{4\pi}{\beta}\max_i \left(\frac{1}{\sqrt{-2f(x_i)}}, \frac{-1}{\delta}\right),$$

if $2\varepsilon A < 1$. Since x_1 and x_2 only are restricted by the condition that $|x_1|$, $|y_1| < Rr$, we get an upper estimate of T that only depends on the same parameters as A, plus δ, but which is independent of ω. Consider now $\chi_T \circ \Phi \circ \chi_{-T}^N \circ \Phi^{-1}$, where Φ is given by proposition 1. This mapping takes the local stable manifold of $J\nabla(H_o + H)$ near $\Phi(x_2, y_2, \mathbf{T}^{d-1}, a)$ into its global unstable manifold. So we must show that the image under this map of the local stable manifold intersect itself. Consider therefore $\Psi = \Phi^{-1} \circ \chi_T \circ \Phi \circ \chi_{-T}^N$ which is the same map expressed in the coordinates given by Φ. We must show that Ψ, which is defined in the complex domain

$$\Omega\colon |x - x_2| + |y - y_2| + |\operatorname{Im}\xi| + |\eta| < \varepsilon^{\frac{1}{2}}.$$

satisfies

$$\Psi(\Omega \cap \Gamma) \cap \Gamma \neq \varnothing,$$

where

$$\Gamma\colon h(x, y + \frac{2}{\beta^2}\langle m, a\rangle) = 0, \xi \in \mathbf{T}^{d-1}, \eta = a.$$

Notice that Ψ is exact symplectic, that

$$|\Psi - id|_\Omega < A_3\varepsilon,$$

and that

$$(H_o + H) \circ \Phi(\Gamma) = (H_o + H) \circ \Phi(\Psi(\Gamma)) = 0.$$

We shall formulate this intersection problem in the coordinates provided by $\tilde{\Phi}$, so let

$$(x_0, 0, \mathbf{T}^{d-1}, a) = \tilde{\Phi}^{-1}(x_2, y_2, \mathbf{T}^{d-1}, a)$$

and

$$\tilde{\Psi} = \tilde{\Phi}^{-1} \circ \Psi \circ \tilde{\Phi}.$$

(This requires that the point (x_2, y_2, ξ, a) belongs to the domain of $\tilde{\Phi}$, and this is an ω-independent condition on x_2.) $\tilde{\Psi}$ is defined in the complex domain

$$\Omega' \colon |x - x_3| + |y| + |\operatorname{Im} \xi| + |\eta| < A_4 \varepsilon^{\frac{1}{2}}$$

and

$$|\tilde{\Psi} - id|_{\Omega'} < A_5 \varepsilon.$$

Moreover $\tilde{\Psi}$ is exact symplectic, being a composition of such transformations, and

$$G(\tilde{\Psi}(\tilde{\Gamma} \cap \Omega')) = G(\tilde{\Gamma} \cap \Omega') = 0,$$

where $\tilde{\Gamma} = \tilde{\Phi}^{-1}(\Gamma)$ and $G = (H_o + H) \circ \Phi \circ \tilde{\Phi}$.

There exists a generating function $\psi(\tilde{x}, y, \tilde{\xi}, \eta)$ defined in

$$|\tilde{x} - x_0)| + |y| + |\eta| < \frac{1}{2} A_4 \varepsilon^{\frac{1}{2}}, \ \tilde{\xi} \in \mathbf{T}^{d-1},$$

such that $\tilde{\Psi}$ is given by

$$\begin{aligned} \tilde{x} &= x - \psi_y, \quad & \tilde{\xi} &= \xi - \psi_\eta \\ y &= \tilde{y} - \psi_{\tilde{x}}, \quad & \eta &= \tilde{\eta} - \psi_{\tilde{\xi}}. \end{aligned}$$

Let now $\eta = a$, $y = 0$, $\tilde{x} = x_0$. By a theorem of Ljusternik-Schnirelmann – see [**15**] for a proof – there exist at least d points $\tilde{\xi}$ such that

$$\psi_{\tilde{\xi}}(x_0, 0, \tilde{\xi}, a) = 0.$$

Hence at such a point we have $\tilde{\eta} = \eta = a$ and we get from $G(\Psi(\Gamma)) = 0$ that $g(x_0 \tilde{y}) = 0$. Since $|\tilde{y}| = |y - \tilde{y}| \sim \varepsilon$ it follows that $x_0 \tilde{y} = 0$ for ε small enough, and since $x_0 \neq 0$, $\tilde{y}$ must be zero. This proves the intersection of the asymptotic manifolds. □

We can easily get transversality of the intersection for ε less than some $\tilde{\varepsilon}_1(H_o)$ if H satisfies some (generic) Poincaré-Melnikov integral condition. The problem is that $\tilde{\varepsilon}_1(H_o)$ tend to become exponentially small with $\frac{1}{|\omega|}$ so we can not apply this to our original problem. But one can easily prove that the invariant hyperbolic torus as well as its local asymptotic manifolds, as given by proposition 1, depend analytically on $|\omega|$, so the distance between the stable and unstable manifolds in, for example, the section $\xi = \xi_0$, $x = \pi$ also depends analytically on $|\omega|$. Since this distance will be $\neq 0$ for some ξ_0 and some $|\omega|$, it must be non-zero for almost all $|\omega|$. So this argument allows us to conclude generic splitting in our original problem. This non constructive argument, however, does not give us transversality of the splitting.

4. Proof of Proposition 1

We define the normal forms $\mathcal{N}(r, \underline{\mu}, \omega)$ as the set of analytic functions N, defined on $B(r)$, of the form

$$N_0 + \langle \omega, \eta \rangle + \kappa xy + M_1 (xy)^2 + 2\langle M_2, \eta \rangle xy + \langle M_3 \eta, \eta \rangle + \mathbf{O}^3(xy, \eta),$$

where N_0 and κ are constants and where $M_i = M_i(x, y, \xi)$ and

$$\begin{aligned} &|\kappa| > \mu_1^{-1}, \\ &|\det[M_3(0,0,\cdot)]| > \mu_2^{-1}, \\ &|N - N_0 - \langle \omega, \cdot \rangle|_{B(r)} < \mu_3. \end{aligned}$$

Here [ ] denotes the meanvalue

$$[g] = \int_{T^{d-1}} g(\xi) d\xi.$$

Proposition 2. *Let $N \in \mathcal{N}(r, \underline{\mu}, \omega)$ and let ω be diophantine with constants K, τ. Then there exists a constant ε_3 such that if*

$$|H|_{B(r)} < \varepsilon < \varepsilon_3,$$

then there exist a constant a and a symplectic transformation $\Phi : B\left(\frac{r}{2}\right) \to B(r)$ such that

$$(N + H) \circ \Phi \in \mathcal{N}\left(\frac{r}{2}, 2\underline{\mu}, \omega\right)$$

and such that $\Phi \circ T_a^{-1}$ is exact, where

$$T_a : (x, y, \xi, \mu) \longmapsto (x, y, \xi, \eta + a),$$

and

$$|\Phi - id|_{B(\frac{r}{2})} + |a| < A\varepsilon.$$

The constants ε_3 and A depend only on $r, \underline{\mu}, K, \tau$.

Proof of proposition 1. Consider now H_o as in proposition 1,

$$H_o = \langle \omega, \eta \rangle + h(x,y) + 2\langle m, \eta \rangle y + \langle M\eta, \eta \rangle.$$

By the transformation $\tilde{\Phi}: B(R_1 r) \to B(r)$ defined in the previous section we get

$$\begin{aligned} H_o \circ \tilde{\Phi} &= \langle \omega, \eta \rangle + g(xy) + \langle M_3 \eta, \eta \rangle \\ &= \langle \omega, \eta \rangle + \kappa xy + M_1 (xy)^2 + \langle M_3 \eta, \eta \rangle + \mathbf{O}^3(xy), \end{aligned}$$

where $\kappa = \alpha\beta$ and $\det M_3 = 2\gamma\beta^{-2}$.

Now we can apply $\Phi \circ T_a^{-1}$ given by proposition 2,

$$(H_o + H) \circ \tilde{\Phi} \circ \Phi \circ T_a^{-1} = \langle \omega, \eta - a \rangle + \kappa' xy + \mathbf{O}^2(xy, \eta - a),$$

module an additive constant. Since $g(xy) = 0$ if and only if $xy = 0$ when r is small enough, independent of ω, the transformed function can be written as

$$\langle \omega, \eta - a \rangle + \frac{\kappa'}{\kappa} g(xy) + \mathbf{O}^2(g(xy), \eta - a).$$

Moreover, using the estimates of Φ one verifies that $|\kappa' - \kappa| < A'\varepsilon$, where A' depends on the same parameters as A. Hence

$$\left| \frac{\kappa'}{\kappa} - 1 \right| < \mu_1 A_1 \varepsilon.$$

Now we consider $\tilde{\Phi}^{-1}: B\left(R_2 \frac{R_1 r}{2}\right) \to B\left(\frac{R_1 r}{2}\right)$. Then

$$\begin{aligned} (H_o + H) \circ \tilde{\Phi} \circ \Phi \circ T_a^{-1} \circ \tilde{\Phi}^{-1} &= \langle \omega, \eta - a \rangle + \frac{\kappa'}{\kappa} h(x, y + \frac{2}{\beta^2} \langle m, \eta \rangle) + \\ &+ \mathbf{O}^2(h(x, y + \frac{2}{\beta^2}\langle m, \eta \rangle), \eta - a) \end{aligned}$$

and proposition 1 is proven, with $R = R_2 \frac{R_1}{2}$. □

5. Proof of Proposition 2

Given an analytic function $F = \Sigma a_{k,l,m}(\xi) x^k y^l \eta^m$ in $B(r)$, we decompose it into its homogeneous components $F = \Sigma (F)_j$ where $(F)_j$ is the sum of all terms in the Taylor series with $\min(k,l)+|m|=j$. Notice that the Poisson bracket $\{F_j, G_k\}$ may not be homogeneous but that

$$(\{F_j, G_k\})_i = 0, \quad i \le j+k-2.$$

We decompose $(F)_j$ further as $(F)_j^0+(F)_j^1+(F)_j^2$ consisting for terms of which $k=l$, $k>l$, $k<l$ respectively. When there is no risk for confusion we shall write F_j^i for $(F)_j^i$. This will be the case for all indexed functions below except for M_1, M_2, M_3 and their tilde-versions.

We now turn to the proof and we first notice that we can assume without restriction that $\mu_i > 1$ and $r < 1$. Notice also that $\tau > d-2$. We first determine a such that

$$2[M_3(0,0,\cdot)]a + [\partial_\eta H(0,0,\dots,0)] = 0.$$

By the assumption on M_3, such an a exists and

$$|a| < c\frac{\mu_2\mu_3^\tau}{r^{2\tau+1}}\varepsilon,$$

– c here and below denotes a constant that only depends on τ. In particular, $T_a\colon B_1 \to B_0$, where $B_j = B(r-j\rho)$, if

$$\varepsilon < c\frac{1}{\mu_2\mu_3^\tau}\rho^{2\tau+2}.$$

We shall take $r = 28\rho$.

Write now $(N+H)\circ T_a = \tilde{N}+\tilde{H}$, $\tilde{N} = N+\langle\omega, a\rangle$. We have $\tilde{H} = F+G$ with

$$\begin{aligned}
F &= 2\langle M_2, a\rangle xy + 2\langle M_3\eta, a\rangle + \partial_\eta\hat{N}(x,y,\xi,\eta)a + H(x,y,\xi,\eta),\\
G &= \langle M_3 a, a\rangle + (\hat{N}(x,y,\xi,\eta+a) - \hat{N}(x,y,\xi,\eta) - \partial_\eta\hat{N}(x,y,\xi,\eta)a)+\\
&\quad + (H(x,y,\xi,\eta+a) - H(x,y,\xi,\eta)),
\end{aligned}$$

where $\hat{N} = N-(N_0+N_1+N_2)$, and

$$\begin{aligned}
|F|_{B_1} &< c\left(\frac{\mu_3}{\rho}|a| + \varepsilon\right) = \varepsilon'\\
|G|_{B_2} &< c\left(\frac{\mu_3}{\rho^2}|a|^2 + \frac{\varepsilon}{\rho}|a|\right) < (\varepsilon')^2.
\end{aligned}$$

Notice that the mean value

$$[\partial_\eta F(0,0,\dots,0)] = 0,$$

by construction of a.

Now we want to determine $S = S_0 + S_1$ and $R = R_0^0 + R_1^0 + \hat{R}$, $\hat{R} \in \mathbf{O}^2(xy, \eta)$, such that

$$\{S, N\} = F - R, \quad [R_0^0 + R_1^0] = R_0^0 + R_1^0,$$

i.e.

$$\begin{aligned}
&\{S_0, N_1\} = F_0 - R_0^0, [R_0^0] = R_0^0 \\
&\{S_1, N_1\} = F_1 - (\{S_0, N_2\})_1 - R_1^0, [R_1^0] = R_1^0 \\
&0 = (F - F_0 - F_1) - \{S_0, N_2 + \hat{N}\} + (\{S_0, N_2\})_1 - \{S_1, N_2 + \hat{N}\} - \hat{R}.
\end{aligned}$$

The *first equation* becomes

$$\begin{aligned}
&\omega \partial_\xi S_0^0 = F_0^0 - R_0^0, R_0^0 = [F(0, 0, \dots, 0)] \\
&\kappa x \partial_x S_0^1 + \omega \partial_\xi S_0^1 = F_0^1 \\
&- \kappa y \partial_x S_0^2 + \omega \partial_\xi S_2^2 = F_0^2.
\end{aligned}$$

The first equation can be solved in Fourier series in ξ and the solution is estimated by standard small divisor estimates ([**15**]). The second equation is expanded in power series in x and each coefficient can be solved in Fourier series in ξ and easily estimated because $\kappa \neq 0$. The third equation is solved in the same way and the outcome is

$$|S_0|_{B_2} < c\frac{\mu_4}{\rho^\tau}\varepsilon', \quad \left|R_0^0\right| < \varepsilon'.$$

where $\mu_4 = \max\left(\mu_1, \frac{1}{K}\right)$.

The *second equation* is of the same kind. We just need to estimate $(\{S_0, N_2\})_1$ by Cauchy estimates in B_3, and we get

$$|S_1|_{B_4} < c\frac{\mu_3\mu_4^2}{\rho^{2\tau+2}}\varepsilon', \quad \left|R_1^0\right|_{B_3} < c\frac{\mu_3\mu_4}{\rho^{\tau+2}}\varepsilon'.$$

Notice that R_1^0 is independent of η by the construction of a.

For the *third equation* we get by a simple Cauchy estimate that

$$\left|\hat{R}\right|_{B_5} < c\frac{\mu_3^2\mu_4^2}{\rho^{2\tau+4}}\varepsilon'.$$

From Cauchy estimates of the derivatives of S in B_5 we get an estimate of the time-t-map Φ^t of $J\nabla S$. It gives for $0 \le t \le 1$

$$\Phi^t: B_7 \to B_6$$
$$\left|\Phi^t - id\right|_{B_6} < c\frac{\mu_3\mu_4^2}{\rho^{2\tau+3}}\varepsilon'$$

if just

$$\varepsilon' < c\frac{\rho^{2\tau+4}}{\mu_3\mu_4^2}.$$

Hence, the transformed Hamiltonian

$$(\tilde{N}+\tilde{H})\circ\Phi^1 - (\tilde{N}+R) = \int_0^1 (\{t\tilde{H}+(1-t)R, S\} + G)\circ\Phi^t dt = \hat{H}$$

is defined in B_7 and verifies

$$\left|\hat{H}\right|_{B_7} < c\frac{\mu_3^3\mu_4^4}{\rho^{4\tau+8}}(\varepsilon')^2.$$

Let's now consider $\tilde{N}+R = N + \langle\omega, a\rangle + R$ which is of the form

$$N_0 + \langle\omega, a\rangle + R_0^0 + \langle\tilde{\omega}, \eta\rangle + \tilde{\kappa}xy + \tilde{M}_1(xy)^2 + \\ + 2\langle\tilde{M}_2, \eta\rangle xy + \langle\tilde{M}_3\eta, \eta\rangle + \mathbf{O}^3(xy, \eta),$$

where $\tilde{\omega}$ and $\tilde{\kappa}$ are constants and $\tilde{\omega} = \omega$ by the choice of a. The difference $\tilde{\kappa} - \kappa$ can be estimated by the second order derivatives of R_1 and $\tilde{M}_3(0,0,\xi) - M_3(0,0,\xi)$ by the second order derivatives of $\hat{R}$. Fix now $1 < \nu < 2$. Then one verifies easily that

$$|\tilde{\kappa}| > \frac{1}{\nu\mu_1},$$
$$\left|\det[\tilde{M}_3(0,0,\cdot)]\right| > \frac{1}{\nu\mu_2}.$$
$$\left|\tilde{N} + R - (N_0 + \langle\omega, a\rangle + R_0) - \langle\omega, \cdot\rangle\right|_{B_8} < \nu\mu_3,$$

if just

$$\varepsilon' < c\frac{\nu - 1}{\mu_2\mu_3^{\tau+2}\mu_4^2}\rho^{4\tau+6}.$$

To summarize this we now abandon the homogeneous notation – different indices will from now on simply denote different functions. Let

$\rho = \frac{r}{28}$, $r_1 = r, r_2 = \frac{3}{4}r$, $\varepsilon_1 = \varepsilon$ and $\nu_2 = \nu$. Then we have constructed a constant a_2 and an exact symplectic mapping Φ_2 such that

$$T_{a_2} \circ \Phi_2 \colon B(r_2) \to B(r_1),$$

and

$$(N + H) \circ T_{a_2} \circ \Phi_2 = N^2 + H^2$$

with

$$N_2 \in \mathcal{N}_1(r_2, \nu_2 \underline{\mu}, \omega),$$
$$\left|H^2\right|_{B(r_2)} < c_0 \frac{\mu_2^2 \mu_3^{2\tau+5} \mu_4^4}{(r_1 - r_2)^{8\tau+12}} \varepsilon_1^2.$$

Moreover

$$|a_2| < c_0 \frac{\mu_2 \mu_3^{\tau}}{(r_1 - r_2)^{2\tau+1}} \varepsilon_1$$

and

$$|T_{a_2} \circ \Phi_2 - id|_{B(r_2)} < c_0 \frac{\mu_2 \mu_3^{\tau+2} \mu_4^2}{(r_1 - r_2)^{4\tau+5}} \varepsilon_1.$$

All this holds under the condition that

$$\varepsilon_1 < \frac{1}{c_0} \frac{\nu - 1}{\mu_2^2 \mu_3^{2\tau+3} \mu_4^2} (r_1 - r_2)^{6\tau+8}.$$

The rest is now an easy calculation. Let

$$r_k = (1 + 2^{-k+1}) \frac{r}{2},$$
$$v_k = 1 + 2^{-k}, \nu_2 \cdots \nu_k \nearrow \nu_\infty \in]1, 2[,$$
$$\underline{\mu}(k+1) = \nu_{k+1} \underline{\mu}(k) = \nu_2 \ldots \nu_{k+1} \underline{\mu},$$
$$\varepsilon_{k+1} = c_0 \frac{\mu_2(k)^2 \mu_3(k)^{2\tau+5} \mu_4(k)^4}{(r_k - r_{k+1})^{8\tau+12}} \varepsilon_k^2, \varepsilon_1 = \varepsilon.$$

In order to continue the above construction infinitely many steps we need that

$$\varepsilon_k < \frac{1}{c_0} \frac{\nu_{k+1} - 1}{\mu_2(k)^2 \mu_3(k)^{2\tau+3} \mu_4(k)^2} (r_k - r_{k+1})^{6\tau+8}.$$

The right hand side of this inequality is larger than BY^k where

$$B = \frac{1}{c_0} \frac{r^{6\tau+8}}{\mu_2^2 \mu_3^{2\tau+3} \mu_4^2} \left(\frac{1}{2}\right)^{8\tau+16}, Y = \left(\frac{1}{2}\right)^{6\tau+9}.$$

If we now let $d_1 = \varepsilon_1$ and $d_{k+1} = AX^k d_k^2$ with

$$A = c_0 \frac{\mu_2^2 \mu_3^{2\tau+5} \mu_4^4}{r^{8\tau+12}} 2^{10\tau+23}, X = 2^{8\tau+12},$$

then $d_k \geq \varepsilon_k$ so it suffices to verify that $d_k < BY^k$ for all k. But this clearly hold if $\varepsilon < BY$ and $\varepsilon < (AX^2)^{-1}$, i.e. if

$$\varepsilon < c_1 \frac{r^{8\tau+12}}{\mu_2^2 \mu_3^{2\tau+5} \mu_4^4}.$$

So under this assumption on ε we can do the above construction infinitely many times and construct a sequence

$$(T_{a_2} \circ \Phi_2) \circ \cdots \circ (T_{a_k} \circ \Phi)_k)$$

with a subsequence converging to a symplectic mapping $\Phi: B\left(\frac{r}{2}\right) \to B(r)$ such that

$$(N + H) \circ \Phi \in \mathcal{N}_1 \left(\frac{r}{2}, 2\underline{\mu}, \omega\right),$$

$$|\Phi - id|_{B(\frac{r}{2})} < c \frac{\mu_2 \mu_3^{\tau+2} \mu_4^2}{r^{4\tau+5}} \varepsilon,$$

$$|a| < c \frac{\mu_2 \mu_3^{\tau}}{r^{2\tau+1}} \varepsilon, a = \Sigma a_k.$$

Φ is not exact but $\Phi \circ T_a^{-1}$ is, because it is the limit of

$$(T_{a_2} \circ \Phi_2 \circ T_{a_2}^{-1}) \circ \cdots \circ (T_{a_2+\cdots+a_k} \circ \Phi_k \circ T_{a_2+\cdots+a_k}^{-1}),$$

which is a composition of exact diffeomorphism, hence exact. This proves proposition 2 and completes the proof of our theorem. □

Note added in proof

The problem of the intersection of the stable and unstable manifolds for hyperbolic tori of arbitrary dimension mentioned in the introduction has been solved by S. Bolotin using quite different variational methods.

Acknowledgement

This work was done while the author was visiting IMPA in Rio de

Janeiro for whose hospitality I am grateful.

References

[1] J. Moser, *Convergent series expansions for quasi-periodic motions.* Mathematische Annalen **169** (1967), 136-176.

[2] S. Graff, *On the conservation of hyperbolic invariant tori for Hamiltonian systems.* Journal of Differential Equations **15** (1974), 1-69.

[3] E. Zehnder, *Generalized implicit function theorems with application to some small divisor problems II.* Communications on Pure and Applied Mathematics **29** (1976), 49-111.

[4] R. de la Llave, C. E. Wayne, *Whiskered and low dimensional tori in nearly integrable Hamiltonian systems.* Manuscript (1989).

[5] V. C. Zelati, I. Ekeland, E. Seré, *A variational approach to homoclinic orbits in Hamiltonian systems.* Mathematische Annalen **288** (1990), 133-160.

[6] T. Cherry, *Asymptotic solutions of analytic Hamiltonian systems.* Journal of Differential Equations **4** (1968), 142-159.

[7] R. McGehee, K. R. Meyer, *Homoclinic points of area preserving diffeomorphisms.* American Journal of Mathematics **96** (1974), 409-421.

[8] P. J. Holmes, J. E. Marsden, *Melnikov's method and Arnold diffusion for perturbations of integrable Hamiltonian systems.* Journal of Mathematical Physics **23** (1982), 669-675.

[9] J. Guckenheimer, P. J. Holmes, *Non-linear oscillations, dynamical systems and bifurcations of vector fields.* Springer, 1983.

[10] E. Zehnder, *Homoclinic points near elliptic fixed points.* Communications on Pure and Applied Mathematics **26** (1973), 131-182.

[11] J. Moser, *A fixed point theorem in symplectic geometry.* Acta Mathematica **141** (1978), 17-34.

[12] J. W. Cassels, *An introduction to the geometry of numbers.* Springer, 1959.

[13] H. Rüssmann, *On optimal estimates of solutions of linear partial differential equations of the first order with constant coefficients on the torus.* Springer Lecture Notes in Physics **38** (1975), 598-624.

[14] H. Rüssmann, *Differentialgleichungen in der Nähe einer Gleichgeunchtslösung.* Mathematische Annalen **169** (167), 55-72.

[15] R. Mañé, *Global variational methods in conservative dynamics.* IMPA, 1991.

L. H. Eliasson
Department of Mathematics
Royal Institute of Technology
S-10044 Stockholm
Sweden